# Infinitude Results for Prime Polynomials

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**Abstract**

At all levels of "algebra" instruction, we acknowledge the similarities between $\mathbb{Z}$ and $\mathbb{F}[x]$, where $\mathbb{F}$ is a field. The key feature in commutative ring theory is that these are *Euclidean integral domains*, possessing a division algorithm. At the completion of eleventh grade intermediate algebra, a successful student will have mastered "long division" for polynomials and integers. Every Euclidean ring is a principal ideal domain, where "prime" and "irreducible" are equivalent; and this in turn implies a generalization of the Fundamental Theorem of Arithmetic – best known in high school algebra by unique factorization of polynomials. Factorization of integers is used in proving Euclid's Theorem about prime infinitude; so how can we expand upon *this* argument in the context of polynomials?

If $\mathbb{F}$ is any field, then the number of prime associate classes in $\mathbb{F}[x]$, each represented by a monic, is $\max\{|\mathbb{F}|, \aleph_0\}$. Unless $\mathbb{F}$ is algebraically closed, $\mathbb{F}[x]$ has nonlinear primes, quadratic or otherwise. The Eisenstein criterion, along with Euclid's Theorem, easily verify that $\mathbb{Q}[x]$ has $\aleph_0$ monic primes of *every* positive degree.

When we want to know how abundantly monic prime quadratics appear in our best-known $\mathbb{F}[x]$ examples, the following fields provide at least $|\mathbb{F}|$ of these $x^2 + Bx + C$ elements. (i) All finite fields except $\mathbb{Z}/(2)$. (ii) Certain subfields of $\mathbb{C}$, such as $\mathbb{Z}(i)$ and $\mathbb{Q}(i)$. (iii) All ordered fields. In particular, the set $\{x^2 + Bx + p : \ B \text{ an integer}, p \text{ a positive prime}\}$ has only two reducible elements in $\mathbb{Q}[x]$, nameld $x^2 \pm (p+1)x + p = (x \pm 1)(x \pm p)$. Fields of rational functions, over an ordered field, are also ordered.

**Keywords:** polynomial ring over a field, prime elements, cardinalities.