ONLY FOR GRADERS

Problem 1 Score	
Problem 2 Score	
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1. Consider the function f defined by

$$f(x) = \begin{cases} Ax + B, & \text{if } \ x < 5 \\ Ax^2 + Bx + 32, & \text{if } \ x \geq 5. \end{cases}$$

   (a) Find an equation satisfied by A and B so that the function f is continuous at $x = 5$.

   (b) Find an equation satisfied by A and B so that the function f is differentiable at $x = 5$.

   (c) Find the values of A and B so that the function f is continuous and differentiable at $x = 5$.

2. Consider the function h given by

$$h(x) = \ln(\ln(\ln(x^2))).$$

   (a) Find the domain of definition of the function h.

   (b) Calculate $h(e)$.

   (c) Calculate $h'(x)$.

   (d) Calculate $h'(e)$.

   (e) Calculate $(h^{-1})'(\ln(\ln(2)))$.

3. Let $k > 0$ with $k \neq 1$ be fixed, and consider

$$F(x) = \int_x^{kx} f(t)\mathrm{d}t, \qquad \text{with } x > 0.$$

Is there a non-constant continuous function $f : (0, +\infty) \to \mathbb{R}$ for which F is a constant function?

4. Find the point on the graph of $y = \sqrt{x}$ that is nearest the point $(p, 0)$ if (i) $p > \frac{1}{2}$; and (ii) $0 < p < \frac{1}{2}$. Express the answer in terms of p.

5. Evaluate the integrals.

(a)

$$\int_0^{\frac{1}{2}} \frac{\sqrt{1-x}}{\sqrt{1+x}} dx.$$

(b)

$$\int \cos(\ln x) dx.$$

6. (a) Use the Mean Value Theorem to show that for all $x > 0$ we have that

$$\frac{1}{x+1} < \ln \frac{x+1}{x} < \frac{1}{x}.$$

(b) Show that the function

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

is monotonic.

(c) Show that for every $x > 0$, $f(x) < e$.

(d) Does $\lim_{x\to\infty} f(x)$ exist? If it does, determine its value. Otherwise, show that it does not exist.

George Mason University

Calculus Olympiad 2024
Solutions Individual Competition

Problem	Steps	Points
1a	The function $f(x)$ is continuous at $x=5$ if $\lim_{x\to 5} f(x) = f(5)$.	2
	Finding the one-sided limits or the limit from the left and $f(5)$ we obtain $5A+B = 25A+5B+32$, or $20A+4B+32=0.$	2
1b	Taking formally the derivative we get that $f'(x) = \begin{cases} A, & \text{if } x<5 \\ 2Ax+B, & \text{if } x \geq 5. \end{cases}$ The function f will be differentiable at 5 if f' is continuous at $x=5$, that is, if $\lim_{x\to 5^-} f'(x) = \lim_{x\to 5^+} f'(x)$. This implies that $A = 10A+B$. This implies that $9A+B=0.$ **Alternatively:** use the limit definition of the derivative at 5 to reach the conclusion.	4
1c	Solving the system $\begin{cases} 20A+4B+32=0 \\ 9A+B=0 \end{cases}$ we get that $A=2 \quad \text{and } B=-18.$	2
2a	We have that $\text{Dom}(h) = \{x\in\mathbb{R}:\ \ln(\ln(x^2))>0\} = \{x\in\mathbb{R}:\ \ln(x^2)>1\}$ $= \{x\in\mathbb{R} : x^2>e\} = (-\infty,-\sqrt{e})\cup(\sqrt{e},+\infty).$	2
2b	We have that $h(e) = \ln(\ln(\ln(e^2))) = \ln(\ln(2)).$	2
2c	We have that $h'(x) = \frac{[\ln(\ln(x^2))]'}{\ln(\ln(x^2))}$	2
	Which is $\frac{[\ln(x^2)]'}{\ln(\ln(x^2))\ln(x^2)}$ $= \frac{2x}{x^2\ln(\ln(x^2))\ln(x^2)} = \frac{2}{x\ln(\ln(x^2))\ln(x^2)}$	2
2d	We have that $h'(e) = \frac{2}{e\ln(\ln(e^2))\ln(e^2)} = \frac{2}{2e\ln(2)} = \frac{1}{e\ln(2)}.$	1
2e	Notice that $(h^{-1})'(\ln(\ln(2))) = \frac{1}{h'(h^{-1}(\ln(\ln(2)))}.$	2
	Since $h(e) = \ln(\ln(2))$ we have that $(h^{-1})'(\ln(\ln(2))) = \frac{1}{h'(e))} = e\ln(2).$	2

Problem	Steps	Points
3	Apply the fundamental theorem of calculus and differentiate the integral to obtain $kf(kx) = f(x)$.	3
	The function $f(x) = \frac{1}{x}$ satisfies the desired properties.	2
4a	A point on the curve $y = \sqrt{x}$ has the form $(x, \sqrt{x})$. Let L be the distance to the point $(p, 0)$. Using the distance formula: $$L = \sqrt{(x-p)^2 + (\sqrt{x} - 0)^2} = \sqrt{(x-p)^2 + x}$$ $$L^2 = (\sqrt{(x-p)^2 + x})^2 = (x-p)^2 + x =$$ $$x^2 - 2px + p^2 + x = x^2 + (1-2p)x + p^2$$	3
	Because $L > 0$, it suffices to minimize L^2. $$\frac{d}{dx}\Big(L^2(x)\Big) = 2x + (1-2p) = 0$$ $$x = -\frac{1}{2}(1-2p) = p - \frac{1}{2}$$	3
	For case (i) Since $p > \frac{1}{2} \Rightarrow p - \frac{1}{2} > 0$, so in the domain of the function, using the sign of the derivative, we obtain that the minimum occurs at $\left(p - \frac{1}{2}, \sqrt{p - \frac{1}{2}}\right)$.	2
	For case (ii) since $0 < p < \frac{1}{2}$, so $1 - 2p > 0$, for $x > 0$, $\dfrac{dL^2}{dx} > 0$, hence L^2 is increasing on the interval $[0, \infty)$ so minimum occurs at $(0, 0)$.	2
5a	Multiply the numerator and the denominator of the integrand by $\sqrt{1-x}$, to rewrite the integral as $$\int_0^{\frac{1}{2}} \frac{1-x}{\sqrt{1-x^2}} dx.$$	2
	Use the trigonometric substitution $x = \sin\theta$ to rewrite the integral as $$\int_0^{\frac{\pi}{6}} \frac{1-\sin\theta}{\cos\theta} \cos\theta d\theta = \int_0^{\frac{\pi}{6}} (1 - \sin\theta) d\theta = (\theta + \cos\theta)\Big\|_0^{\frac{\pi}{6}} =$$ $$\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1$$	3
5b	Use the substitution $u = \ln x$ to rewrite the integral as $\int e^u \cos u du$.	2
	Use integration by parts twice to find $\int e^u \cos u du = \frac{1}{2}e^u(\sin u + \cos u) + C = \frac{1}{2}x(\sin(\ln x) + \cos(\ln x)) + C$.	3

Problem	Steps	Points
6a	Let $a > 0$. Apply the Mean Value Theorem to $f(x) = \ln x$ on the interval $[a, a+1]$ to show that $\ln(a+1) - \ln a = \frac{1}{c}$ for some $c \in (a, a+1)$.	3
	Use the fact that $\frac{1}{a+1} < \frac{1}{c} < \frac{1}{a}$ to obtain that $\frac{1}{a+1} < \ln \frac{a+1}{a} = \ln(1 + \frac{1}{a}) < \frac{1}{a}$. Since a is an arbitrary positive number, this shows that the inequality $\frac{1}{x+1} < \ln \frac{x+1}{x} = \ln(1 + \frac{1}{x}) < \frac{1}{x}$ for all $x > 0$.	2
6b	Find the derivative $$f'(x) = e^{x \ln(1+\frac{1}{x})}[\ln(1 + \frac{1}{x}) - \frac{1}{x+1}]$$	3
	Use part a) to conclude that $f'(x) > 0$.	2
6c	Take ln of both sides to show that the given inequality is equivalent to $\ln(1 + \frac{1}{x}) < \frac{1}{x}$.	3
	Use part a) to conclude that $f(x) < e$ for all $x > 0$.	2
6d	Rewrite the function as $$f'(x) = e^{x \ln(1+\frac{1}{x})}.$$	2
	Use l'Hôpital's rule to find that the limit of the exponent as x goes to infinity is equal to 1, and so, the required limit is e.	3