ONLY FOR GRADERS

Problem 1 Score	
Problem 2 Score	
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Total Score	

1. (a) Use the half-angle identities ($\cos^2 \frac{\theta}{2} = \frac{1 + \cos\theta}{2}$ and $\sin^2 \frac{\theta}{2} = \frac{1 - \cos\theta}{2}$) to evaluate

$$\int \frac{1}{1 + \cos(x + a)} dx,$$

where a is a constant.

(b) Use part (a) to evaluate

$$\int \frac{1}{1 - \sin x} dx.$$

2. Projectile Dynamics Optimization:

   The path of a projectile fired at an angle θ (where $0 \leq \theta \leq \pi/2$) and initial speed v_0 from a point $(0, y_0)$ in the x-y plane where $y_0 > 0$, can be expressed in terms of the equations

$$
\begin{aligned}
x(t) &= (v_0 \cos\theta)t \\
y(t) &= -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + y_0
\end{aligned}
$$

   Here $x(t)$ represents the horizontal distance the projectile travels and $y(t)$ represents the height of the projectile above ground level ($y = 0$) as a function of time t. The projectile starts at horizontal position $x = 0$ and is shot from an initial height above the ground y_0 (when $t = 0$).

   Identify the optimal value of θ such that the horizontal distance traveled by the projectile before it hits the ground ($y = 0$) is maximized. Please list your corresponding value of θ, the projectile range (i.e. what is the initial angle and how far away does the projectile land?), and explain how you arrived at your result.

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that f' and f'' are also continuous on $[0,1]$.

   a. Show that if the series
   $$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$$
   is convergent, then $f(0) = 0$ and $f'(0) = 0$.

   b. Conversely, show that if $f(0) = 0$ and $f'(0) = 0$, then
   $$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$$
   is convergent.

   (**Hint:** You may consider using the Mean Value Theorem.)

George Mason University

Calculus Olympiad 2024
Solutions Team Competition

Problem	Steps	Poi
1a	Rewrite the integral as $$\int \frac{1}{2\cos^2\frac{x+a}{2}}dx.$$	3
	Use the substitution $u = \frac{x+a}{2}$ to rewrite the integral as $$\int \frac{1}{\cos^2 u}du = \int \sec^2 u du = \tan u + C = \tan\frac{x+a}{2} + C$$	2
1b	Use $\sin x = -\cos\left(x + \frac{\pi}{2}\right)$ to rewrite the integral as $\int \frac{1}{1+\cos\left(x+\frac{\pi}{2}\right)}dx$.	3
	Use part a) to conclude that $\int \frac{1}{1-\sin x}dx = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) + C$	2
2	Substitute in for t and set $y = 0$ (1) $$0 = -\left(\frac{g}{2v_0^2\cos^2\theta}\right)x^2 + (\tan\theta)x + y_0$$ or (2) $$0 = -\frac{1}{2}gx^2 + \left(v_0^2\cos\theta\sin\theta\right)x + y_0v_0^2\cos^2\theta$$ This determines $x = x(\theta)$ at the point where $y = 0$.	2
	Now compute $dx/d\theta$ (3) $$0 = -\frac{gx}{v_0^2\cos^2\theta}\frac{dx}{d\theta} - \left(\frac{gx^2}{v_0^2}\right)\frac{\sin\theta}{\cos^3\theta} + \tan\theta\frac{dx}{d\theta} + \frac{1}{\cos^2\theta}x,$$ or (4) $$-gx\frac{dx}{d\theta} + v_0^2\left[\cos^2\theta - \sin^2\theta\right]x + v_0^2\sin\theta\cos\theta\frac{dx}{d\theta} - 2y_0v_0^2\cos\theta\sin\theta$$	3
	Set $dx/d\theta = 0$ to get and cancel constant factors to get (5) $$0 = -\frac{gx_{\text{opt}}^2}{v_0^2}\frac{\sin\theta_{\text{opt}}}{\cos^3\theta_{\text{opt}}} + \frac{1}{\cos^2\theta_{\text{opt}}}x_{\text{opt}}.$$ or (6) $$0 = \cos 2\theta_{\text{opt}}x_{\text{opt}} - y_0\sin 2\theta_{\text{opt}}$$ where we have used $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.	2
	Then, the optimal distance is (7) $$x_{\text{opt}} = \frac{v_0^2}{g\tan\theta_{\text{opt}}}.$$ or equivalently (8) $$x_{\text{opt}} = y_0\tan 2\theta_{\text{opt}}$$ Putting either or these values of x back in equation (1) or (2) allows one to find the optimal θ which has (9) $$\sin\theta_{\text{opt}} = \frac{1}{\sqrt{2\left(1 + \frac{gy_0}{v_0^2}\right)}}$$ The value of any trig function of θ would work.	3

Problem	Steps	Points
3a	Suppose that $f(0) \neq 0$ and wlog suppose $f(0) > c$. Then, there exist $N > 0$ such that $f(x) \geq c/2$ and $x \in [0, 1/N]$ so that $$+\infty > \sum_{n=1}^{N-1} f\left(\frac{1}{n}\right) + \sum_{n=N}^{\infty} f\left(\frac{1}{n}\right) \geq \sum_{n=1}^{N-1} f\left(\frac{1}{n}\right) + \sum_{n=N}^{\infty} \frac{c}{2} = +\infty,$$ a contradiction. **Alternatively,** since the series is convergent, we have that $$\lim_{n\to\infty} f(\frac{1}{n}) = 0.$$ On the other hand, since f is continuous at 0 and $\{\frac{1}{n}\}$ converges to 0, we have that $$\lim_{n\to\infty} f(\frac{1}{n}) = f(\lim_{n\to\infty} \frac{1}{n}) = f(0) = 0.$$	3
	Suppose that $f'(0) \neq 0$ and wlog suppose $f'(0) > c$. The application of the mean value theorem to f on $[0, 1/n]$ leads to $f(1/n) - f(0) = f(1/n) = f'(e_n)/n$ where $0 \leq e_n \leq 1$ and $\lim_{n\to\infty} e_n = 0$. Then, there exist $N \in \mathbb{N}$ such that $f(e_n) \geq c/2$ for $n \geq N$ so that $$+\infty > \sum_{n=1}^{\infty} f\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{f'(e_n)}{n}$$ $$= \sum_{n=1}^{N-1} \frac{f'(e_n)}{n} + \sum_{n=N}^{\infty} \frac{f'(e_n)}{n} \geq \sum_{n=1}^{N-1} \frac{f'(e_n)}{n} + \frac{c}{2} \sum_{n=N}^{\infty} \frac{1}{n} = +\infty,$$ a contradiction. **Alternatively,** Suppose that $f'(0) \neq 0$ and wlog suppose $f'(0) = c > 0$. The application of the mean value theorem to f on $[0, 1/n]$ leads to $f(1/n) - f(0) = f(1/n) = f'(e_n)/n$ where $0 \leq e_n \leq 1$ and $\lim_{n\to\infty} e_n = 0$. Then $$\lim_{n\to\infty} \frac{\frac{f'(e_n)}{n}}{\frac{1}{n}} = c > 0,$$ thus, by the Limit Comparison Test, the given series has the same nature as the harmonic series and, therefore, is divergent, which contradicts the hypothesis.	3

Problem	Steps	Points
3b	Apply the mean value theorem to f on $[0,1/n]$ to obtain $f(1/n) = f(1/n) - f(0) = f'(e_n)/n$ for $e_n \in (0,1/n)$, and again to f' on $[0,e_n]$ to obtain $f'(e_n) = f'(e_n) - f'(0) = f''(d_n)e_n$ where $d_n \in (0,e_n)$. Hence $$\sum_{n=1}^{\infty}\left\| f\left(\frac{1}{n}\right)\right\| = \sum_{n=1}^{\infty}\|f''(d_n)\|\frac{e_n}{n} = \sum_{n=1}^{\infty}\|f''(d_n)\|\frac{1}{n^2} \leq \left(\sup_{x\in[0,1]}\|f''(x)\|\right)\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty.$$ Since absolute summability implies summability, the result follows. (The last part of the proof could be stated as follows: Let $M = \sup_{x\in[0,1]}\|f''(x)\|$. Then for all n, $$\|f''(d_n)\|\frac{e_n}{n^2} \leq \frac{M}{n^2}.$$ Since the series $\sum_{n=1}^{\infty}\frac{M}{n^2}$ is convergent, by the Direct Comparison Test, we have that the series $$\sum_{n=1}^{\infty}\|f''(d_n)\|\frac{e_n}{n}$$ is convergent. Thus, the given series is absolutely convergent and, therefore, convergent.)	4