

A One-Million Point Steiner Tree

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Abstract

The Steiner tree problem is one of the oldest optimization problems in all of mathematics, with roots going back to the 1600’s and even to the ancient Greeks. Given a set of points in the plane, it seeks an arrangement of line segments connecting the points together with minimum total length (a distance metric must be specified, commonly Euclidean (L_2) or rectilinear (L_1)).

The GeoSteiner software by Warme, Winter and Zachariasen has long been the fastest unified software for solving these problems. It uses geometric processing to construct an edge-weighted hypergraph $H = (V, E)$, then computes a minimum spanning tree in H using branch-and-cut to solve an integer program having an exponential number of “subtour elimination constraints” (“subtours” for short). This approach is entirely analogous to the best methods for the closely-related traveling salesman problem.

Computational speed of such cutting-plane methods crucially depends on the choice of cutting plane inequalities used. Inequalities that yield solutions quickly are called “strong;” those that do not are called “weak.” Unfortunately, “strength” of an inequality is an ethereal property lacking any good formal definition, and is not well understood. Methods for quantitatively estimating the strength of inequalities are therefore very desirable.

We present two such “strength indicators”: extreme point ratio (EPR) and centroid distance (CD), for which we have obtained closed forms regarding the subtour inequalities of the spanning tree in hypergraph polytope used in GeoSteiner. These indicators corroborate prior computational experience with these inequalities. They also reveal that subtours of large cardinality could be substantially *stronger* than the corresponding subtours of small cardinality, a property previously neither known nor suspected.

Exploiting this knowledge within GeoSteiner has yielded a 10-fold increase in the size of instances solved, culminating in an optimal solution to a Euclidean Steiner tree instance on 1,000,000 random points uniformly distributed in the unit square. This has been called “the largest classic NP-hard problem ever solved.” A large poster of this Steiner tree solution will be displayed at the talk.

Keywords: Steiner tree problem, integer program, branch-and-cut, strength of inequalities, hypergraph, spanning tree, NP-hard, enumerative combinatorics, geometry.