Math 664 : Linear Algebra with Data Applications

- **Prerequisites:** Equivalent of MATH322 and basic programming experience. The course assumes a mathematically rigorous background in linear algebra (vector spaces, linear transformations, eigenspaces) and ability to write advanced mathematical proofs. Projects will require basic programming/computational experience.
- **Instructor:** Tyrus Berry, tberry@gmu.edu, http://math.gmu.edu/~berry/
- **Office:** Blackboard Collaborate
- **Office hours:** TBA, see details below.
- **Course Website:** Blackboard, https://mymasonportal.gmu.edu/
- **Book:** Linear Algebra in Action (Second Edition) by Harry Dym
- **Classroom:** Asynchronous online

Overview

The course covers the theoretical linear algebra required for advanced mathematics courses and data analysis applications. Key abstract concepts include fundamental subspaces, duality, invariants, and the operator algebra. These are grounded in their respective linear algebra instances, namely, bases, adjoints, eigenspaces, and matrix decompositions including the Jordan decomposition and SVD. Topics are presented at a level of abstraction that allows natural generalization. Analytic assignments explore connections to related advanced mathematics topics. Computational projects connect the abstract concepts to foundational methods and techniques in data science such as dimensionality reduction, graph embeddings, and regularized regression.

Students will work individually or in pairs on a report and presentation that explores how linear algebra generalizes to a primary field of study in mathematics, potential topics include:

- Higher Order Linear Algebra: Tensor Decompositions (HOSVD and CP)
- Analysis: The Riesz Representation Theorem
- Analysis: The Fredholm Alternative
- Analysis: The Spectral Theorem
- Topology: Sard's Theorem and the Regular Value Theorem
- Foundations: Duality and Naturality in Category Theory
- Algebra: Witt's Theorem and bilinear forms
- Algebra: Wedderburn principal theorem and Jordan-Chevalley decomposition
- Analysis/Algebra/Geometry: Lie Group-Lie Algebra connection, Classification

A report should develop and explain the necessary definitions, give examples, and explain the key results and how they generalize results in linear algebra as well as how they differ.

Office Hours

Office hours will all be held online via Blackboard Collaborate and will alternate every other week between mandatory individual (one-on-one with professor) 15 minute meetings and optional (come as you need to) group office hours. Surveys will be emailed out at the beginning of the semester to schedule times.

Student Engagement: Weekly Learning Plan

Your week should be split up into three study periods (on three different days) each consisting of:

1. Viewing a short lecture video on Blackboard.
2. Working a quick problem on Blackboard (unlimited attempts).
3. Reading a section or two in the book.
4. Working out details or exercises from the book (your choice).

Details and exercises worked from the book should be maintained in a journal that will be collected periodically. The remaining time will be devoted to projects. Projects will have a large computational aspect and Matlab/Octave/FreeMat are highly recommended but alternatives may be acceptable, please discuss in advance if you would like to use an alternative.

Grading

Grades will be based on four components:

1. 40% - Computational projects
2. 20% - Special Topic Report/Presentation
3. 20% - Journal of exercises
4. 20% - Blackboard quizzes

Weekly Schedule (book sections in parentheses)

1. (1.1-1.3) Introduction, review of basic linear algebra (vector spaces, linear transformations, bases)
2. (1.4-1.5,4.1) Dual spaces and transposes, isomorphisms and coordinates, subspaces and triangular matrices
3. (2.1-2.3) Dual space isomorphism, the matrix algebra, Gaussian elimination
4. (2.4-2.8) Conservation of dimension, matrices and dimension, regression and bases
5. (4.2-4.6) Matrix decompositions (overview), invariant subspaces, diagonalizability
6. (4.7-4.11) Generalized eigenvectors, Jordan chains, and Jordan canonical form
7. (4.12-4.13) Symmetry and the Singular Value Decomposition (SVD), Uniqueness caveats, implications for PCA
8. (9.1,10.1) Metrics and inner products and Multi-Dimensional Scaling (MDS), Determinant, trace, and norm
9. (5.1-5.5) Matrix norms, operator norms, Kernel PCA and Kernel Regression
10. (7.1-7.3,8.1-8.2,11.1-11.2) Inner products, quadratic forms, regularization and ridge regression/kernel ridge
11. (7.6-7.7,13.1,13.3-13.4,13.8) Matrix square roots, Matrix exponentials and dynamical systems, Linear stability analysis
12. (18.1,18.5,18.7) Intro to filtering and optimal control
13. (16.1,16.7) Tensors, tensor norms, multilinear forms, duality in optimization and Support Vector Machines (SVM)
14. (notes) Selection of optional topics and applications

Learning Outcomes

Students will gain **advanced knowledge, including rigorous mathematical construction,** of the fundamental subspaces (quotient, product, direct sum, invariant), Gaussian elimination, orthogonal matrices, pseudo-inverse, linear maps/operators, determinant and trace, eigenvalues and eigenvectors, dual spaces, Rayleigh quotients, Jordan canonical form, the spectral theorem, singular vectors and singular values (SVD), matrix exponentials, matrix square roots, tensor products, and multilinear forms. Students will be able to explain intuitively and derive rigorously the connection of these abstract concept to the key applications of Principal Component Analysis (PCA), regularized least squares regression, MultiDimensional Scaling (MDS), and Support Vector Machines (SVM). **Students should be able to derive and program these methods without references.**

Students will gain a **working knowledge** of abstract mathematical concepts including duality (including duals spaces, maps, and related isomorphisms), the structure of linear spaces, and the structure of the operator algebra. **Students will be prepared to generalize these concepts to other areas of mathematics such as abstract algebra and linear analysis.**

Students will gain an **deep familiarity** with a selection of the following optional topics and applications.

Optional topics: Restricted conjugacy and algebra of orbits (e.g. SO(n)); linear mappings on normed linear spaces; dual extremal problems; introduction to numerical linear algebra; introduction to linear programming; dual spaces of derivatives and forms in multivariable calculus; spectral radius; algebra of resolvents; matrices over other fields.

Optional applications: Kernel methods, sparsity and compression, FFT, dynamics, linear stability analysis, Markov processes, state estimation, optimal control, graph/network analysis, dimensionality reduction, clustering, regularized regression, data visualization, feature identification. Computational and analytical assignments are given.

Academic Policies

Mason is an Honor Code university; please see the Office for Academic Integrity for a full description of the code and the honor committee process. With collaborative work, names of all the participants should appear on the work. Collaborative projects may be divided up so that individual group members complete portions of the whole, provided that group members take sufficient steps to ensure that the pieces conceptually fit together in the end product. Other projects are designed to be undertaken independently. In the latter case, you may discuss your ideas with others and conference with peers on drafts of the work; however, it is not appropriate to give your paper to someone else to revise. You are responsible for making certain that there is no question that the work you hand in is your own. If only your name appears on an assignment, your professor has the right to expect that you have done the work yourself, fully and independently.

Disability Accommodations

Disability Services at George Mason University is committed to upholding the letter and spirit of the laws that ensure equal treatment of people with disabilities. Under the administration of University Life, Disability Services implements and coordinates reasonable accommodations and disability-related services that afford equal access to university programs and activities. Students can begin the registration process with Disability Services at any time during their enrollment at George Mason University. If you are seeking accommodations, please visit http://ds.gmu.edu/ for detailed information about the Disability Services registration process. Disability Services is located in Student Union Building I (SUB I), Suite 2500. Email: ods@gmu.edu
Phone: (703) 993-2474

Non-Discrimination Policy

George Mason University is committed to providing equal opportunity and an educational and work environment free from any discrimination on the basis of race, color, religion, national origin, sex, disability, veteran status, sexual orientation, gender identity, gender expression, age, marital status, pregnancy status or genetic information. George Mason University shall adhere to all applicable state and federal equal opportunity/affirmative action statutes and regulations.

The University is dedicated to ensuring access, fairness and equity for minorities, women, individuals with disabilities, and veterans (as covered by law) in its educational programs, related activities and employment. George Mason University shall thus maintain a continuing affirmative action program to identify and eliminate discriminatory practices in every phase of university operations.

Any employee who becomes aware of sexual harassment or other potentially discriminatory behavior must contact Compliance, Diversity, and Ethics.

Retaliation against an individual who has raised claims of illegal discrimination or has cooperated with an investigation of such claims is prohibited.