Olympiad 2023
Practice Problems

1. Find the sum of the real numbers a and b for which

$$\lim_{x\to\infty} (\sqrt{x^2+x+1}+ax) = b.$$

(Hint: For what value of a is this limit finite?)

2. Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} x^3 \sin(1/x) \ , & x \neq 0 \\ 0 \ , & x = 0 \ . \end{cases}$$

Show that $f(x)$ is differentiable at 0 and find $f'(0)$.
(Hint: Use the limit definition for the derivative.)

3. Write an equation for the tangent line to the graph of $f(x) = \left(\dfrac{x+1}{x-1}\right)^2$ at the point at which $x = 0$.

4. Show that

$$\frac{d}{dx}\left(\ln \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right) = \frac{2}{x\sqrt{1-x^4}}.$$

(Hint: Use properties of the logarithm to simplify your work.)

5. Find $y'(0)$, given that $e^y + xy = e$.

6. Let $f(x) = \dfrac{\ln(1+x)}{x}$, for $x > 0$.

   (a) Show that $f(x)$ is strictly decreasing on $(0, \infty)$.

   (b) Show that $f(x) < 1$ for $x > 0$.

7. Let $f(x) = x\sqrt{3-2x}$ for $x < \dfrac{3}{2}$. Find the real numbers a, b, c for which the function $F(x) = (ax^2 + bx + c)\sqrt{3-2x}$ is an antiderivative for $f(x)$ for $x < \dfrac{3}{2}$.
   (Hint: It is easier if you use integration rather than differentiation.)

8. Find the critical points and identify the relative and the absolute extrema for the function defined by $f(x) = \displaystyle\int_0^x e^{t^2}(t^3 - 3t + 2)dt$, for $x \in R$.
   (Hint: You will need the Fundamental Theorem of Calculus.)

9. Find
$$\lim_{x\to\infty}\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x.$$
   (Hint: L'Hôpital's rule)

10. Let the function $f(x)$ be defined by
$$f(x) = \begin{cases} ax^2 \ , & x \le 1 \\ \ln x + x \ , & x > 1 \ , \end{cases}$$
   where a is a constant for which $f(x)$ is continuous at 1. Find the area of the plane region bounded by the curves $y = f(x)$, $y = 0$, $x = 0$, and $x = e$.

11. Evaluate $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} \cos(k/n)$.
(Hint: Riemann sums)

12. Evaluate $\int \frac{3\cos x}{\sin^2 x - \sin x - 2} dx$.

13. Evaluate
$$\int x^5 \sqrt{1 - x^3} dx.$$

14. (a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin(2x)}{1 + \cos^2 x} dx$.
    (b) Let $f(x)$ be a function that is continuous on $[0, 1]$. Prove that
$$\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx.$$
    (Hint: Use $x = \pi - u$.)
    (c) Use the result from part (b) to evaluate $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$.

15. Determine whether the integral $\int_0^{\infty} \frac{1}{x(x^2 + 1)} dx$ is convergent and find its value in case of convergence.

16. Let S be a sphere of center O and radius R, let P be a point on S. Find the volume of the spherical cap formed when we cut off the sphere S by the plane α which is perpendicular to the radius OP and intersects it at the point H, where $OH = h$, $0 < h < R$.

17. Let the sequence I_n be defined as follows:

$$I_0 = \int_0^1 e^{-x} dx$$

$$I_n = \int_0^1 x^n e^{-x} dx.$$

(a) Find I_0.

(b) Use integration by parts to show that

$$I_n = -\frac{1}{e} + nI_{n-1}.$$

(c) Show that for $x \in [0, 1]$,

$$\frac{x^n}{e} \leq x^n e^{-x} \leq x^n.$$

(d) Integrate the inequality above to show that

$$\frac{1}{e(n+1)} \leq I_n \leq \frac{1}{n+1}.$$

(e) Using the reduction formula for I_n, we can show that

$$I_n = \frac{n!}{e}[e - (1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!})].$$

How can you use the results from the previous parts to find

$$\lim_{n\to\infty} I_n$$

and

$$\lim_{n\to\infty} (1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!})?$$

18. Let $f(x) = x \tan^{-1} x$ and $g(x) = \ln(1 + x^2)$. Prove that for every $x \in [0, 1]$, $f(x) \geq g(x)$ and then find the area of the region bounded by $y = f(x)$, $y = g(x)$, $x = 0$, and $x = 1$.
(Hint: To prove the given inequality, consider the difference of the two functions and study the sign of its first and second derivatives on the given interval.)